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SCALAR INVARIANTS OF TWO LINEAR VECTOR FUNCTIONS.

By CHARLES J. JOLY, M.A., Fellow of Trinity College, Dublin.

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XVIII.

SCALAR INVARIANTS OF TWO LINEAR VECTOR FUNCTIONS. By CHARLES J. JOLY, M.A., Fellow of Trinity College, Dublin.

[Read December 9, 1895.]

1.—*Introduction.*

It is well known that the coefficients of $\phi^3 - m_1 \phi^2 + m_2 \phi - m_3 = 0$, the symbolical cubic of a linear vector function ϕ, may be expressed in the form,

$$m_1 = \frac{S(\phi\alpha\beta\gamma + \alpha\phi\beta\gamma + \alpha\beta\phi\gamma)}{S\alpha\beta\gamma},$$

$$m_2 = \frac{S(\alpha\phi\beta\phi\gamma + \phi\alpha\beta\phi\gamma + \phi\alpha\phi\beta\gamma)}{S\alpha\beta\gamma},$$

and

$$m_3 = \frac{S(\phi\alpha\phi\beta\phi\gamma)}{S\alpha\beta\gamma}.$$

These coefficients are independent of the arbitrary vectors α, β, and γ, and on this account they have been called *Invariants* by Hamilton.

Scalar invariants of a similar type, depending on two or more linear vector functions, are considered in the present Paper. In addition to the six primary invariants of two functions, five independent invariants have been found. These are called l_2, l_3, l_3', M_2 and $N_3 - N_3'$. The last is skew, and though doubtless its square can be expressed in the terms of the six primary invariants, it must be reckoned as distinct.

A comparatively simple method leads to what may be called "reducing systems" of invariants. By the aid of one of these systems a series of invariants may, without difficulty, be reduced to the eleven fundamental

invariants, and many invariants are in this Paper thus reduced. Many other invariants have been reduced by this method, but it is hoped that enough has been shown to facilitate reduction in most if not in all cases. In Article 25 (p. 728), some hint is given of invariants of more than two functions, and certain differential operators are noticed.

2.—*Explanation of Notation.*

In treating the invariants of the two functions ϕ and θ, it seems to be most convenient to express them in terms of the roots and axes of one of the functions. Thus, if ρ_1, ρ_2, and ρ_3 are unit vectors along the axes of ϕ, and if g_1, g_2, and g_3 are the corresponding roots, the function θ is completely defined by the three equations,

$$\theta\rho_1 = a_{11}\rho_1 + a_{12}\rho_2 + a_{13}\rho_3\,; \quad \theta\rho_2 = a_{21}\rho_1 + a_{22}\rho_2 + a_{23}\rho_3\,; \quad \theta\rho_3 = a_{31}\rho_1 + a_{32}\rho_2 + a_{33}\rho_3.$$

The nine coefficients a_{12}, a_{21}, &c., are, in general, distinct.

If the minors of the determinant, $m_3' = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, are denoted by the symbols $A_{11} = a_{22}a_{33} - a_{23}a_{32}$; $A_{23} = a_{12}a_{31} - a_{11}a_{32}$; $A_{32} = a_{21}a_{13} - a_{11}a_{23}$; and six other symbols with double suffixes which follow from symmetry,

$$\begin{aligned} V\theta\rho_2\theta\rho_3 &= A_{11}V\rho_2\rho_3 + A_{12}V\rho_3\rho_1 + A_{13}V\rho_1\rho_2, \\ V\theta\rho_3\theta\rho_1 &= A_{21}V\rho_2\rho_3 + A_{22}V\rho_3\rho_1 + A_{23}V\rho_1\rho_2, \\ V\theta\rho_1\theta\rho_2 &= A_{31}V\rho_2\rho_3 + A_{32}V\rho_3\rho_1 + A_{33}V\rho_1\rho_2. \end{aligned}$$

Again,

$$\theta^2\rho_1 = b_{11}\rho_1 + b_{12}\rho_2 + b_{13}\rho_3\,; \quad \theta^2\rho_2 = b_{21}\rho_1 + b_{22}\rho_2 + b_{23}\rho_3\,; \quad \theta^2\rho_3 = b_{31}\rho_1 + b_{32}\rho_2 + b_{33}\rho_3,$$

if $b_{11} = a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31}$; $b_{23} = a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33}$; $b_{32} = a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32}$,

and if the other symbols have values which follow from symmetry.

Similarly, the vector products $V\theta^2\rho_2\theta^2\rho_3 = B_{11}V\rho_2\rho_3 + B_{12}V\rho_3\rho_1 + B_{13}V\rho_1\rho_2$ define the symbols B,

$$B_{11} = b_{22}b_{33} - b_{23}b_{32}\,; \qquad B_{23} = b_{12}b_{31} - b_{11}b_{32}\,; \qquad B_{32} = b_{21}b_{13} - b_{11}b_{23}.$$

3.—*Establishment of Useful Identities.*

In order not to interrupt future discussions, some connexions among the symbols are here given.

Thus,
$$b_{11} = a_{11}^2 + (a_{11}a_{22} - A_{33}) + (a_{11}a_{33} - A_{22}),$$
$$= a_{11}\Sigma a_{11} - (A_{22} + A_{33});$$

and two similar formulæ give b_{11}, b_{22}, and b_{33} in terms of a_{11}, a_{22}, a_{33}, and A_{11}, A_{22}, and A_{33}.

The determinant of § 2 becomes on expansion

$$m_3' = a_{23}a_{31}a_{12} + a_{32}a_{13}a_{21} + a_{11}a_{22}a_{33} - \Sigma a_{11}a_{23}a_{32}.$$

Now

$$\Sigma a_{11}a_{23}a_{32} = \Sigma a_{11}(a_{22}a_{33} - A_{11}) = 3\Pi a_{11} - \Sigma a_{11}A_{11}, \text{ if } \Pi a_{11} = a_{11}a_{22}a_{33};$$

accordingly,
$$a_{23}a_{31}a_{12} + a_{32}a_{13}a_{21} = m_3' + 2\Pi a_{11} - \Sigma a_{11}A_{11}.$$

Again,
$$a_{23}A_{23} + a_{32}A_{32} = a_{23}a_{31}a_{12} + a_{32}a_{13}a_{21} - 2a_{11}a_{23}a_{32}.$$
$$= m_3' - \Sigma a_{11}A_{11} + 2a_{11}A_{11}.$$

Also,
$$B_{11} = \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} & a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} \\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} & a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = A_{11}A_{11} + A_{12}A_{21} + A_{13}A_{31},$$

and
$$B_{23} = A_{21}A_{13} + A_{22}A_{23} + A_{23}A_{33}$$

express the coefficients B in terms of the coefficients A.

Further, g_1, g_2, and g_3 being roots of $g^3 - m_1g^2 + m_2g - m_3 = 0$,

$$g_1^2 - m_1g_1 + m_2 = g_2g_3, \quad \text{and} \quad g_1^2 + g_2^2 + g_3^2 = m_1^2 - 2m_2, \text{ and also}$$

$$g_2^2g_3^2 = m_3g_1 - m_1m_3 + m_2g_2g_3 = m_2g_1^2 - (m_1m_2 - m_3)g_1 + (m_2^2 - m_1m_3).$$

4.—*Expressions for the Scalar invariants of a single function, ϕ or θ.*

If the cubic of ϕ is $\phi^3 - m_1\phi^2 + m_2\phi - m_3 = 0$, and that of θ is $\theta^3 - m_1'\theta^2 + m_2'\theta - m_3' = 0$,

$$m_1' = \frac{S\,\theta\alpha\beta\gamma + S\,\theta\beta\gamma\alpha + S\,\theta\gamma\alpha\beta}{S\,\alpha\beta\gamma},$$

$$m_2' = \frac{S\,\alpha\theta\beta\theta\gamma + S\,\beta\theta\gamma\theta\alpha + S\,\gamma\theta\alpha\theta\beta}{S\,\alpha\beta\gamma},$$

and $$m_3' = \frac{S\,\theta\alpha\theta\beta\theta\gamma}{S\,\alpha\beta\gamma},$$ where α, β, and γ are *any* three vectors which are not coplanar. If in particular the vectors are taken to be ρ_1, ρ_2, and ρ_3, the axes of ϕ,

$$m_1' = a_{11} + a_{22} + a_{33}\,; \quad m_2' = A_{11} + A_{22} + A_{33}, \text{ and } m_3' = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix};$$

using the formulæ of the second Article.

Also, $m_1 = g_1 + g_2 + g_3$; $m_2 = g_2g_3 + g_3g_1 + g_1g_2$, and $m_3 = g_1g_2g_3$; g_1, g_2, and g_3 being the roots of the cubic of ϕ.

5.—*Derivation of three new invariants of two functions, ϕ and θ.*

Again, if $\theta + u\phi$ satisfy the equation, $\psi^3 - m_1''\psi^2 + m_2''\psi - m_3'' = 0$, where u is a scalar and $\psi = \theta + u\phi$,

$$m_1'' = m_1' + um_1\,; \quad m_2'' = m_2' + ul_2 + u^2m_2\,; \quad m_3'' = m_3' + ul_3' + u^2l_3 + u^3m_3\,;$$

verifying for the values zero and infinity given to u.

l_2, l_3', and l_3 are new invariants of the functions θ and ϕ, since m_1'', m_2'', and m_3'' are independent of α, β, and γ, and u is arbitrary.

Now, $$l_2 = \frac{\Sigma\, S\,\alpha(\theta\beta\phi\gamma + \phi\beta\theta\gamma)}{S\,\alpha\beta\gamma} = \frac{\Sigma\, S\,\theta\alpha(\phi\beta\gamma + \beta\phi\gamma)}{S\,\alpha\beta\gamma},$$

or, on replacing α, β, and γ by ρ_1, ρ_2, and ρ_3, and remembering that $\phi\rho_1 = g_1\rho_1$,

$$l_2 = \Sigma(g_2 + g_3)\frac{S\,\theta\rho_1\rho_2\rho_3}{S\,\rho_1\rho_2\rho_3} = \Sigma\, a_{11}(g_2 + g_3).$$

In like manner,

$$l_3 = \frac{\Sigma S\,\theta\alpha\,\phi\beta\,\phi\gamma}{S\,\alpha\beta\gamma} = \Sigma g_2 g_3 \frac{S\,\theta\rho_1\rho_2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma a_{11} g_2 g_3,$$

$$l_3' = \frac{\Sigma S\,\phi\alpha\,\theta\beta\,\theta\gamma}{S\,\alpha\beta\gamma} = \Sigma g_1 \frac{S\,\rho_1\theta\rho_2\theta\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma A_{11} g_1,$$

using the formulæ of the second Article.

6.—*First reducing system of invariants.*

The three invariants

$$m_1' = \Sigma a_{11}; \quad m_1 m_1' - l_2 = \Sigma a_{11} g_1, \text{ and } l_3 - m_1 l_2 + (m_1^2 - m_2) m_1' = \Sigma a_{11} g_1^2$$

form what may be called a reducing system.

If any invariant is found to be of the form $\Sigma a_{11} f(g_1)$, where $f(g_1)$ is any rational and integral function, it may be reduced to a sum of the three invariants multiplied by functions of m_1, m_2, and m_3. For, by a simple process of division,

$$f(g) = F(g)(g^3 - m_1 g^2 + m_2 g - m_3) + Pg^2 + Qg + R;$$

and if g_1 is a root of the cubic, $f(g_1) = Pg_1^2 + Qg_1 + R$, P, Q, and R being functions of m_1, m_2, and m_3.

Thus, any invariant of the type $\dfrac{\Sigma S\,\phi^{n_1}\theta\phi^{n_2}\alpha\phi^{n_3}\beta\phi^{n_4}\gamma}{S\,\alpha\beta\gamma}$ is reducible, n_1, n_3, n_3, and n_4 being integers. On putting ρ_1, ρ_2, ρ_3, for α, β, and γ, the invariant is at once reduced to $\Sigma g_1^{n_1+n_2}(g_2^{n_3} g_3^{n_4} + g_2^{n_4} g_3^{n_3})\, a_{11}$. Now, a rational integral function symmetrical in g_2 and g_3 may easily be reduced to a function of g_1. A typical term is of the form

$$(g_2 g_3)^n (g_2^{n'} + g_3^{n'}) = (g_1^2 - m_1 g_1 + m_2)^n (\Sigma g_1^{n'} - g_1^{n'}).$$

7.—*Determination of a fourth new invariant and a second reducing system.*

The functions $\theta\phi$ and $\phi\theta$ have the same invariants.

$(\theta\phi)^3 - M_1(\theta\phi)^2 + M_2(\theta\phi) - M_3 = 0$, changes into

$(\phi\theta)^3 - M_1(\phi\theta)^2 + M_2(\phi\theta) - M_3 = 0$, when multiplied *by* ϕ and *into* ϕ^{-1}.

Here $$M_1 = \frac{\Sigma S\,\theta\phi\rho_1\rho_2\rho_3}{S\,\rho_1\rho_2\rho_3} = \Sigma\, g_1 a_{11},$$

$$M_2 = \frac{\Sigma\, S\rho_1\,\theta\phi\rho_2\,\theta\phi\rho_3}{S\,\rho_1\rho_2\rho_3} = \Sigma\, g_2 g_3 A_{11} = \Sigma\, g_2 g_3 \frac{S\rho_1\,\theta\rho_2\,\theta\rho_3}{S\,\rho_1\rho_2\rho_3},$$

and $$M_3 = \frac{S\,\theta\phi\rho_1\,\theta\phi\rho_2\,\theta\phi\rho_3}{S\,\rho_1\rho_2\rho_3} = g_1 g_2 g_3 m_3' = m_3 m_3'.$$

Thus a new invariant M_2, symmetrical in θ and ϕ is found, and this combined with $m_2' = \Sigma A_{11}$ and $l_3' = \Sigma A_{11} g_1$ gives an invariant $\Sigma A_{11} g_1^2$. These three form a new reducing system by means of which any invariant of the type $\Sigma A_{11} f(g_1)$ may be reduced. In terms of the other invariants

$$\Sigma A_{11} g_1^2 = \Sigma A_{11}(g_2 g_3 + m_1 g_1 - m_2) = M_2 + m_1 l_3' - m_2 m_2'.$$

Any invariant of the form $\dfrac{\Sigma\, S\,\phi^{n_1}\alpha\,\phi^{n_2}\theta\phi^{n_3}\beta\,\phi^{n_4}\theta\phi^{n_5}\gamma}{S\,\alpha\beta\gamma}$ is reducible by this system.

8.—*Reducing system linear in the square of one function.*

A reducing system may also be found depending on θ^2. The invariant

$$\frac{\Sigma\, S\,\theta^2\alpha\beta\gamma}{S\,\alpha\beta\gamma} = (m_1'^2 - 2m_2') = h_1^2 + h_2^2 + h_3^2,$$

if h_1, h_2, and h_3 are the roots of θ. This may be expressed by means of the symbols b_{11}, b_{22}, and b_{33} of Art. 2, since $m_1'^2 - 2m_2' = \dfrac{\Sigma\, S\,\theta^2\rho_1\rho_2\rho_3}{S\,\rho_1\rho_2\rho_3} = \Sigma\, b_{11}$.

Now the l_2 invariant of θ^2 and ϕ is

$$\frac{\Sigma\, S\,\theta^2\rho_1(\phi\,\rho_2\rho_3 + \rho_2\phi\rho_3)}{S\,\rho_1\rho_2\rho_3} = \Sigma\, b_{11}(g_2 + g_3),$$

and the l_3 invariant of the same functions is

$$\frac{\Sigma\, S\,\theta^2\rho_1\,\phi\rho_2\,\phi\rho_3}{S\,\rho_1\rho_2\rho_3} = \Sigma\, g_2 g_3\, b_{11}.$$

Since by Art. 2, $b_{11} = a_{11}\,\Sigma a_{11} + A_{11} - \Sigma\, A_{11} = A_{11} + a_{11} m_1' - m_2'$, this new reducing system is merely a combination of the two former systems. By its means any invariant linear in θ^2 may be reduced.

9.—*Reducing system quadratic in the square of one function.*

A fourth reducing system may also be found, viz. ΣB_{11}, $\Sigma B_{11}g_1$, and $\Sigma B_{11}g_1^2$. Referring to the second Article for the definition of these symbols,

$$\frac{\Sigma S\rho_1\theta^2\rho_2\theta^2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma B_{11}\,; \quad \frac{\Sigma S\phi\rho_1\theta^2\rho_2\theta^2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma B_{11}g_1\,;$$

and

$$\frac{\Sigma S\phi^2\rho_1\theta^2\rho_2\theta^2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma B_{11}g_1^2,$$

are those invariants. The first is the m_2 of θ^2, the second is the l_3' of θ^2 and ϕ, and the third the l_3' of θ^2 and ϕ^2.

The values of these invariants, in terms of the old, may in several ways be assigned.

$$\frac{\Sigma S\rho_1\phi^2\rho_2\phi^2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma g_2^2g_3^2 = m_2^2 - 2m_1m_3\,;$$

hence, changing ϕ to θ,

$$\Sigma B_{11} = m_2'^2 - 2m_1'm_3'.$$

$$\frac{\Sigma S\theta\rho_1\phi^2\rho_2\phi^2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma a_{11}g_2^2g_3^2 = m_3\Sigma a_{11}g_1 - m_1m_3m_1' + m_2\Sigma a_{11}g_2g_3$$
$$= m_2l_3 - m_3l_2.$$

So $$\Sigma B_{11}g_1 = m_2'l_3' - m_3'l_2.$$

$$\frac{\Sigma S\theta^2\rho_1\phi^2\rho_2\phi^2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma b_{11}g_2^2g_3^2 = m_3\Sigma b_{11}g_1 - m_1m_3\Sigma b_{11} + m_2\Sigma b_{11}g_2g_3\,;$$

but $$\Sigma b_{11} = m_1'^2 - 2m_2'\,; \quad \Sigma b_{11}g_1 = m_1'\Sigma a_{11}g_1 + \Sigma A_{11}g_1 - m_2'm_1$$
$$= l_3' - m_1'l_2 + m_1(m_1'^2 - m_2'), \text{ and}$$

$$\Sigma b_{11}g_2g_3 = m_1'\Sigma a_{11}g_2g_3 + \Sigma A_{11}g_2g_3 - m_2'm_2$$
$$= M_2 + m_1'l_3 - m_2'm_2. \qquad \text{(See Arts. 5 and 7).}$$

Consequently,

$$\Sigma b_{11}g_2^2g_3^2 = m_2M_2 + m_2m_1'l_3 + m_3l_3' - m_3m_1'l_2 + m_2'(m_3m_1 - m_2^2)\,;$$

and, on interchange of θ and ϕ,

$$\Sigma B_{11}g_1^2 = m_2'M_2 + m_2'm_1l_3' + m_3'l_3 - m_3'm_1l_2 + m_2(m_3'm_1' - m_2'^2).$$

10.—*Reducing system linear in $\theta\phi\theta$.*

Other reducing systems may easily be derived. Take for instance the function $\theta\phi\theta$.

$$\begin{aligned}\theta\phi\theta\rho_1 &= \theta(g_1a_{11}\rho_1 + g_2a_{12}\rho_2 + g_3a_{13}\rho_3)\\ &= (g_1a_{11}a_{11} + g_2a_{12}a_{21} + g_3a_{13}a_{31})\rho_1\\ &\qquad + (g_1a_{11}a_{12} + g_2a_{12}a_{22} + g_3a_{13}a_{32})\rho_2\\ &\qquad\qquad + (g_1a_{11}a_{13} + g_2a_{12}a_{23} + g_3a_{13}a_{33})\rho_3\\ &= c_{11}\rho_1 + c_{12}\rho_2 + c_{13}\rho_3.\end{aligned}$$

$$\frac{\Sigma S\phi\theta\phi\rho_1\rho_2\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma a_{11}g_1^2 = l_3 - m_1l_2 + (m_1^2 - m_2)m_1' \qquad \text{by Art. 6, and}$$

changing θ and ϕ, the invariant

$$\Sigma c_{11} = l_3' - m_1'l_2 + (m_1'^2 - m_2')m_1 \quad \text{results.}$$

$$\begin{aligned}\Sigma c_{11}g_1 &= \Sigma(g_1a_{11}^2 + g_2a_{12}a_{21} + g_3a_{13}a_{31})g_1 = \Sigma a_{11}^2g_1^2 + 2\Sigma g_2g_3\,a_{23}a_{32}\\ &= (\Sigma a_{11}g_1)^2 - 2\Sigma g_2g_3A_{11}, \quad \text{as} \quad A_{11} = a_{22}a_{33} - a_{23}a_{32};\end{aligned}$$

and

$$\begin{aligned}\Sigma c_{11}g_1^2 &= \Sigma a_{11}^2g_1^3 + \Sigma g_2g_3(g_2+g_3)\,a_{23}a_{32},\\ &= \Sigma a_{11}g_1\,\Sigma a_{11}g_1^2 - \Sigma A_{11}g_2g_3(g_2+g_3),\end{aligned}$$

and this reducing system is consequently expressed in terms of known invariants.

11.—*General remarks on reducing systems.*

More generally, if ψ is any linear vector function, and if

$$\psi\rho_1 = c_{11}\rho_1 + c_{12}\rho_2 + c_{13}\rho_3,$$

and if two similar expressions hold for $\psi\rho_2$ and $\psi\rho_3$,

$$m_1(\psi) = \frac{\Sigma S\psi\alpha\beta\gamma}{S\alpha\beta\gamma} = \Sigma c_{11},$$

$$l_2(\psi,\phi) = \frac{\Sigma S\psi\alpha(\beta\phi\gamma + \phi\beta\gamma)}{S\alpha\beta\gamma} = \Sigma(g_2+g_3)c_{11},$$

and

$$l_2(\psi,\phi^2) = \frac{\Sigma S\psi\alpha(\beta\phi^2\gamma + \phi^2\beta\gamma)}{S\alpha\beta\gamma} = \Sigma(g_2^2+g_3^2)c_{11},$$

form a reducing system for invariants of the type $\Sigma c_{11}f(g_1)$. $l_2(\psi,\phi)$ denotes the l_2 invariant formed from ψ and ϕ.

In like manner, a reducing system is formed by $m_1(\psi)$,

$$l_2(\psi, \theta) = \frac{\Sigma\, S\alpha(\psi\beta\theta\gamma + \theta\beta\psi\gamma)}{S\,\alpha\beta\gamma},$$

and
$$l_2(\psi, \theta^2) = \frac{\Sigma\, S\alpha(\psi\beta\theta^2\gamma + \theta^2\beta\psi\gamma)}{S\,\alpha\beta\gamma}.$$

Now,
$$V\psi\rho_2\theta\rho_3 = (c_{22}a_{33} - c_{23}a_{32})V\rho_2\rho_3 + (c_{23}a_{31} - c_{21}a_{33})V\rho_3\rho_1 + (c_{21}a_{32} - c_{22}a_{31})V\rho_1\rho_2,$$

with a similar result for $V\theta\rho_2\psi\rho_3$, found by interchanging c and a.

So
$$l_2(\psi, \theta) = \Sigma(c_{22}a_{33} + c_{33}a_{22}) - \Sigma(c_{23}a_{32} + c_{32}a_{23}),$$

and
$$l_2(\psi, \theta^2) = \Sigma(c_{22}b_{33} + c_{33}b_{22}) - \Sigma(c_{23}b_{32} + c_{32}b_{23}).$$

Or, more simply,
$$m_1(\psi) = \Sigma\, c_{11},$$
$$m_1(\psi)m_1' - l_2(\psi, \theta) = \Sigma\, a_{11}c_{11} + \Sigma(c_{23}a_{32} + c_{32}a_{23}),$$

and
$$m_1(\psi)(m_1'^2 - 2m_2') - l_2(\psi, \theta^2) = \Sigma\, b_{11}c_{11} + \Sigma(c_{23}b_{32} + c_{32}b_{23}),$$

form a reducing system.

Again, forming the invariant

$$\frac{\Sigma\, S\psi\alpha(\phi\beta\theta\gamma + \theta\beta\phi\gamma)}{S\,\alpha\beta\gamma}, \quad \text{or} \quad \frac{\Sigma\, S\phi\rho_1(\psi\rho_2\theta\rho_3 + \theta\rho_3\psi\rho_2)}{S\,\rho_1\rho_2\rho_3},$$

which may be called $l_3(\psi, \theta, \phi)$,

$$l_3(\psi, \theta, \phi) = \Sigma\, g_1[c_{22}a_{33} + c_{33}a_{22} - c_{23}a_{32} - c_{32}a_{23}]. \quad \text{Also,}$$
$$l_3(\psi, \theta, \phi^2) = \Sigma\, g_1^2[c_{22}a_{33} + c_{33}a_{22} - c_{23}a_{32} - c_{32}a_{23}].$$

These invariants, $l_2(\psi, \theta)$, $l_3(\psi, \theta, \phi)$, $l_3(\psi, \theta, \phi^2)$, form a new reducing system.

12.—*Derivation of reducing systems from a given system.*

Given any reducing system of the type

$$\Sigma\, c_{11} = I_1, \quad \Sigma\, c_{11}g_1 = I_2, \quad \Sigma\, c_{11}g_1^2 = I_3; \quad \text{on solution}$$
$$c_{11}(g_1 - g_2)(g_1 - g_3) = c_{11}(3g_1^2 - 2m_1g_1 + m_2) = I_1g_2g_3 - I_2(g_2 + g_3) + I_3.$$

Hence, multiplying both sides by $(3g_2^2 - 2m_1g_2 + m_2)(3g_3^2 - 2m_1g_3 + m_3)$, and dividing by the product of the three expressions $3g_1^2 - 2mg_1 + m_3$ (that

is, by the discriminant of the cubic in g), $c_{11} = J_1 g_1^2 + J_2 g_1 + J_3$, where J_1, J_2, and J_3 are invariants whose values in terms of I_1, I_2, and I_3 are easily found.

Any function of c_{11}, c_{22}, and c_{33}, symmetrical in c_{22} and c_{33}, can be reduced to the form,

$$f_{11} = K_1 g_1^2 + K_2 g_1 + K_3,$$

K_1, K_2, and K_3 being invariants if the coefficients of the rational function f_{11} are invariants. Hence follows the reducing system,

$$\Sigma f_{11} = K_1 \Sigma g_1^2 + K_2 \Sigma g_1 + 3 K_3,$$
$$\Sigma f_{11} g_1 = K_1 \Sigma g_1^3 + K_2 \Sigma g_1^2 + K_3 \Sigma g,$$
$$\Sigma f_{11} g_1^2 = K_1 \Sigma g_1^4 + K_2 \Sigma g_1^3 + K_3 \Sigma g_1^2.$$

Further, if I_1, I_2, I_3, and the coefficients of f_{11}, are reducible to the fundamental invariants, so are these new functions.

13.—*Disguised reducible forms. First example.*

Reducible forms may sometimes present themselves in a disguised manner. Suppose, for instance, any two reducing systems are given in the form

$$\Sigma b_{11} = I_1; \qquad \Sigma b_{11} g_1 = I_2; \qquad \Sigma b_{11} g_1^2 = I_3, \text{ and}$$
$$\Sigma c_{11} = J_1; \qquad \Sigma c_{11} g_1 = J_2, \text{ and } \Sigma c_{11} g_1^2 = J_3.$$

The general invariant linear in the b's and c's is

$$I_1(X_1 J_1 + X_2 J_2 + X_3 J_3) + I_2(X_4 J_1 + X_5 J_2 + X_6 J_3)$$
$$+ I_3(X_7 J_1 + X_8 J_2 + X_9 J_3).$$

If this involves the b's and c's only in the combinations $b_{11}c_{11}$, $b_{22}c_{22}$, and $b_{33}c_{33}$, on expressing that the coefficients of $b_{22}c_{33}$ and of $b_{33}c_{22}$ are zero, the two equations,

$$(X_1 + X_2 g_3 + X_3 g_3^2) + g_2(X_4 + X_5 g_3 + X_6 g_3^2) + g_2^2(X_7 + X_8 g_3 + X_9 g_3^2) = 0,$$

and

$$(X_1 + X_2 g_2 + X_3 g_2^2) + g_3(X_4 + X_5 g_2 + X_6 g_2^2) + g_3^2(X_7 + X_8 g_2 + X_9 g_2^2) = 0,$$

are the result.

Subtracting one from the other, and dividing by $g_2 - g_3$,

$$(X_2 - X_4) + (g_2 + g_3)(X_3 - X_7) + g_2 g_3 (X_6 - X_8) = 0.$$

This cannot hold for the other pairs of roots unless $X_2 = X_4$, $X_3 = X_7$, and $X_6 = X_8$. The vanishing coefficients give, therefore, the equation,

$$X_1 + X_2(g_2 + g_3) + X_3(g_2^2 + g_3^2) + X_5 g_2 g_3 + X_6 g_2 g_3 (g_2 + g_3) + X_9 g_2^2 g_3^2 = 0.$$

Reducing this again to a quadratic in g_1, its coefficients must vanish if it is satisfied by the three roots.

Easy reduction and solution for X_1, X_2, and X_3 give,

$$\begin{aligned} X_1 &= X_5 m_2 + X_6(m_1 m_2 + m_3) + X_9 m_2^2, \\ X_2 &= -X_5 m_1 - X_6 m_1^2 - X_9(m_1 m_2 - m_3), \\ \text{and} \quad X_3 &= X_5 + X_6 m_1 + X_9 m_2. \end{aligned}$$

Now, X_5, X_6, and X_9 are arbitrary, so the three invariants,

$$\begin{aligned} K_1 &= m_2 I_1 J_1 - m_1(I_1 J_2 + I_2 J_1) + I_1 J_3 + I_3 J_1 + I_2 J_2, \\ K_2' &= (m_1 m_2 + m_3) I_1 J_1 - m_1^2(I_1 J_2 + I_2 J_1) + m_1(I_1 J_3 + I_3 J_1) + I_2 J_3 + I_3 J_2, \\ K_3' &= m_2^2 I_1 J_1 - (m_1 m_2 - m_3)(I_1 J_2 + I_2 J_1) + m_2(I_1 J_3 + I_3 J_1) + I_3 J_3, \end{aligned}$$

satisfy the conditions.

The coefficient of $b_{11} c_{11}$ in K_1 is $3g_1^2 - 2m_1 g_1 + m_2$; this being the result of substituting a root of the cubic in its first derived, is equal to $(g_1 - g_2)(g_1 - g_3)$.

In K_2' the coefficient is

$$\begin{aligned} 2g_1^3 + 2m_1 g_1^2 - 2m_1^2 g_1 + m_1 m_2 + m_3 &= 3g_1^3 + m_1 g_1^2 + (m_2 - 2m_1^2) g_1 + m_1 m_2, \\ &= (g_1 + m_1)(3g_1^2 - 2m_1 g_1 + m_2); \end{aligned}$$

and similarly for K_3'

$$\begin{aligned} g_1^4 + 2m_2 g_1^2 - 2(m_1 m_2 - m_3) g_1 + m_2^2 &= g_1^4 - m_2 g_1^2 + 2m_3 g_1 + m_2(3g_1^2 - 2m_1 g_1 + m_2), \\ &= (g_1^2 + m_2)(g_1 - g_2)(g_1 - g_3). \end{aligned}$$

Hence, subtracting m_1K_1 from K_2' and m_2K_1 from K_3'

$$K_1 = m_2I_1J_1 - m_1(I_1J_2 + I_2J_1) + I_1J_3 + I_3J_1 + I_2J_2 = \Sigma\, b_{11}c_{11}(g_1 - g_2)(g_1 - g_3),$$

$$K_2 = m_3I_1J_1 - m_1I_2J_2 + I_2J_3 + I_3J_2 \qquad = \Sigma\, b_{11}c_{11}g_1(g_1 - g_2)(g_1 - g_3),$$

and $K_3 = m_3(I_1J_2 + I_2J_1) - m_2I_2J_2 + I_3J_3 \qquad = \Sigma\, b_{11}c_{11}g_1^2(g_1 - g_2)(g_1 - g_3),$

which is a reducing system.

14.—*Second Example.*

As a further example of disguised reducible functions, let the coefficients of $b_{22}c_{33}$ and $b_{33}c_{22}$ be equal in the function of the last Article.

Exactly as before, $X_2 = X_4$, $X_3 = X_7$, and $X_6 = X_8$. Suppose, moreover, that the coefficients of $b_{11}c_{11}$, $b_{22}c_{22}$, and $b_{33}c_{33}$ vanish,

$$X_1 + 2X_2g_1 + (2X_3 + X_5)g_1^2 + 2X_6g_1^3 + X_9g_1^4 = 0,$$

or reduced to a cubic,

$$X_1 + (2X_2 + m_3X_9)g_1 + (2X_3 + X_5 - m_2X_9)g_1^2 + (2X_6 + m_1X_9)g_1^3 = 0.$$

Identifying this with the cubic $m_3 - m_2g_1 + m_1g_1^2 - g_1^3 = 0$, if

$$X_1 = Xm_3\,; \quad 2X_2 + m_3X_9 = Xm_2\,; \quad 2X_3 + X_5 - m_2X_9 = Xm_1, \text{ and}$$
$$2X_6 + m_1X_9 = -X,$$

on substitution, the following combinations are found:

$$L_1 = I_1J_3 + I_3J_1 - 2I_2J_2 = \Sigma(b_{22}c_{33} + b_{33}c_{22})(g_2 - g_3)^2\,;$$

$$2m_3I_1J_1 - m_2(I_1J_2 + I_2J_1) + m_1(I_1J_3 + I_3J_1) - (I_2J_3 + I_3J_2)$$
$$= \Sigma\,(b_{22}c_{33} + b_{33}c_{22})(g_2 + g_3)(g_2 - g_3)^2\,;$$

and $$-m_3(I_1J_2 + I_2J_1) + m_2(I_1J_3 + I_3J_1) - m_1(I_2J_3 + I_3J_2) + 2I_3J_3$$
$$= \Sigma(b_{22}c_{33} + b_{33}c_{22})g_1(g_2 + g_3)(g_2 - g_3)^2.$$

Subtracting the second of these from m_1 times the first,

$$L_2 = -2m_3I_1J_1 + m_2(I_1J_2 + I_2J_1) - 2m_1I_2J_2 + I_2J_3 + I_3J_2$$
$$= \Sigma(b_{22}c_{33} + b_{33}c_{22})g_1(g_2 - g_3)^2,$$

and, subtracting the third from m_1 times L_2,

$$L_3 = -2m_1m_3 I_1J_1 + (m_1m_2 + m_3)(I_1J_2 + I_2J_1) - 2m_1^2 I_2J_2 - m_2(I_1J_3 + I_3J_1)$$
$$+ 2m_1(I_2J_3 + I_3J_2) - 2I_3J_3 = \Sigma(b_{22}c_{33} + b_{33}c_{22})g_1^2(g_2 - g_3)^2,$$

which also form a reducing system.

15.—*Functions of the second order. Reduction of invariants linear in one of these.*

The general reduction of invariants may be systematically discussed by forming linear vector functions in all possible ways from θ and ϕ.

The functions of the second order are θ^2, $\theta\phi$, $\phi\theta$, and ϕ^2. The invariants l_2, l_3, and l_3', arising from any function of the second order (ψ_2 say) with θ and ϕ (these being of the first order may generally be denoted by ψ_1), are all reducible, and most of them have been reduced.

The invariants $l_3(\psi_2, \theta, \phi)$ that call for any consideration are $l_3(\theta\phi, \theta, \phi)$ and $l_3(\phi\theta, \theta, \phi)$, which latter can be deduced from the former by interchange of θ and ϕ. Now,

$$l_3(\theta\phi, \theta, \phi) = \frac{\Sigma S\phi\rho_1(\theta\phi\rho_2\theta\rho_3 + \theta\rho_2\theta\phi\rho_3)}{S\rho_1\rho_2\rho_3} = \Sigma g_1(g_2 + g_3)A_{11},$$

and this is reducible.

16.—*Reduction of invariants of two functions of the second order.*

The l_2 invariants of the functions of the second order are obviously reducible with perhaps the exception of $l_2(\theta\phi, \phi\theta)$. But this invariant is also reducible, for the coefficient of $V\rho_2\rho_3$ in

$V\phi\theta\rho_2\theta\phi\rho_3$ is $g_2g_3 a_{22}a_{33} - g_3^2 a_{32}a_{23}$, and in

$V\theta\phi\rho_2\phi\theta\rho_3$ it is $g_2g_3 a_{22}a_{33} - g_2^2 a_{32}a_{23}$. Hence the invariant is

$$2\Sigma g_2g_3 a_{22}a_{33} - \Sigma(g_2^2 + g_3^2)(a_{22}a_{33} - A_{11}) = \Sigma A_{11}(g_2^2 + g_3^2) + (\Sigma a_{11}g_1)^2 - \Sigma a_{11}\Sigma a_{11}g_1^2.$$

The l_3 invariants, found by combining the same function ψ_2 twice with ψ_1, are obviously reducible.

Combining two different functions, ψ_2 and ψ_2', with ψ_1, so as to form the invariants $l_3(\psi_2, \psi_2', \psi_1)$, the invariant whose reduction is least obvious is perhaps $l_3(\theta\phi, \phi\theta, \phi)$. But this is, by the present Article,

$$\Sigma g_1[2g_2g_3a_{22}a_{33} - (g_2^2 + g_3^2)(a_{22}a_{33} - A_{11})]$$
$$= 2m_3(\Sigma a_{11})^2 - \Sigma a_{11}g_1 \Sigma a_{11}g_2g_3 + \Sigma A_{11}g_1(g_2^2 + g_3^2).$$

Since the coefficient of $V\rho_2\rho_3$ in $V(\theta\phi\rho_2\theta\rho_3 + \theta\rho_2\theta\phi\rho_3)$ is $(g_2 + g_3)A_{11}$, $l_3(\phi^2, \theta\phi, \theta)$ is reducible. Also, the coefficient of $V\rho_2\rho_3$ in $V(\phi\theta\rho_2\theta\rho_3 + \theta\rho_2\phi\theta\rho_3)$

is $$(g_2a_{22}a_{33} - g_3a_{23}a_{32}) + (g_3a_{22}a_{33} - g_2a_{23}a_{32}) = (g_2 + g_3)A_{11},$$

and the invariant $l_3(\phi^2, \phi\theta, \theta)$ is the same as $l_3(\phi^2, \theta\phi, \theta)$.

17.—*Two irreducible invariants, N_3 and N_3'.*

In the last Article, invariants quadratically involving the functions ψ were all considered; the invariants involving them cubically will be noticed here.

The l_3 invariants involving θ^2 or ϕ^2 twice are at once reducible, and so are those involving $\theta\phi$ or $\phi\theta$ twice, and either θ^2 or ϕ^2 once. $l_3(\phi^2, \theta\phi, \phi\theta)$ is, by the relation in the last Article, $\Sigma g_1^2[A_{11}(g_2^2 + g_3^2) - a_{22}a_{33}(g_2 - g_3)^2]$, and this is reducible by the L_3 function of Art. 14, simplified by putting $b = c = a$. Only the two invariants, $l_3(\theta\phi, \theta\phi, \phi\theta)$ and $l_3(\phi\theta, \phi\theta, \theta\phi)$ remain for consideration.

These two invariants may be expressed as the sum of three determinants,

$$\frac{\Sigma S\phi\theta\rho_1\theta\phi\rho_2\theta\phi\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma g_2g_3 \begin{vmatrix} g_1a_{11} & g_2a_{12} & g_3a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = N_3;$$

and, on slight modification,

$$\frac{\Sigma S\theta\phi\rho_1\phi\theta\rho_2\phi\theta\rho_3}{S\rho_1\rho_2\rho_3} = \Sigma g_1 \begin{vmatrix} g_2g_3a_{11} & g_3g_1a_{12} & g_1g_2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = N_3'.$$

18.—*Reduction of their sum,* $N_3 + N_3'$.

It is at once apparent that these invariants are of a new type, and combining them by addition and subtraction, some simplification is effected,

$$N_3 + N_3' = \Sigma \begin{vmatrix} 2g_1g_2g_3a_{11} & g_3(g_1^2+g_2^2)a_{12} & g_2(g_1^2+g_3^2)a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2m_3\Sigma a_{11}A_{11} + \Sigma(a_{23}A_{23} + a_{32}A_{32})g_1(g_2^2+g_3^2)$$

$$= m_3'(m_1m_2 - 3m_3) - \Sigma a_{11}A_{11}(m_1m_2 - 5m_3) + 2\Sigma A_{11}a_{11}g_1(g_2^2+g_3^2),$$

using the formula $a_{23}A_{23} + a_{32}A_{32} = m_3' - \Sigma a_{11}A_{11} + 2a_{11}A_{11}$ of Art. 3 and the relation $\Sigma g_1(g_2^2+g_3^2) = m_1m_2 - 3m_3$.

Now, $m_1m_2 - 5m_3 - 2g_1(g_2^2+g_3^2)$

$$= [g_1^2(g_2+g_3) + g_1(g_2+g_3)^2 + g_1g_2g_3 + g_2g_3(g_2+g_3)] - 5g_1g_2g_3 - 2g_1(g_2^2+g_3^2)$$

$$= (g_2+g_3)(g_1-g_2)(g_1-g_3).$$

So $N_3 + N_3' = m_3'(m_1m_2 - 3m_3) - \Sigma a_{11}A_{11}(g_2+g_3)(g_1-g_2)(g_1-g_3)$. This is reducible by the K functions of Art. 13, and its value is

$$N_3 + N_3' = -3m_3m_3' + m_3m_1'm_2' + m_3'm_1m_2 + l_2m_2m_2' - l_3m_1m_2' - l_3'm_1'm_2 + l_3l_3' - l_2M_2.$$

19.—*Their difference is a new skew invariant,* $N_3 - N_3'$.

The difference of these invariants is

$$N_3' - N_3 = \Sigma \begin{vmatrix} 0 & (g_1^2-g_2^2)g_3a_{12} & -(g_3^2-g_1^2)g_2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \Sigma g_1(g_2^2-g_3^2)(a_{23}A_{23} - a_{32}A_{32}).$$

Now, $$a_{23}A_{23} - a_{32}A_{32} = a_{23}(a_{31}a_{12} - a_{11}a_{32}) - a_{32}(a_{13}a_{21} - a_{11}a_{23})$$

$$= a_{23}a_{31}a_{12} - a_{32}a_{13}a_{21}.$$

Hence the invariant is

$$N_3' - N_3 = (g_2 - g_3)(g_3 - g_1)(g_1 - g_2)(a_{23}a_{31}a_{12} - a_{32}a_{13}a_{21}).$$

This invariant is of a distinct type from any previously found, and is irreducible. Moreover, since N_3 is changed into N_3' when θ and ϕ are interchanged, it may be concluded that $a_{23}a_{31}a_{12} - a_{32}a_{13}a_{21}$ is proportional to the product of the differences of the roots of θ.

20.—*First reducing system depending on the skew invariant.*

If $D_{11} = \begin{vmatrix} g_1 a_{11} & g_2 a_{12} & g_3 a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, the invariant $N_3 = \Sigma D_{11} g_2 g_3$ belongs to a reducing system ΣD_{11}, $\Sigma D_{11} g_1$, and $\Sigma D_{11} g_1^2$.

$$\Sigma D_{11} = \frac{\Sigma S\phi\theta\rho_1\theta\rho_2\theta\rho_3}{S\rho_1\rho_2\rho_3} = \frac{\Sigma S\phi\theta\rho_1\theta\rho_2\theta\rho_3}{S\theta\rho_1\theta\rho_2\theta\rho_3} \cdot \frac{S\theta\rho_1\theta\rho_2\theta\rho_3}{S\rho_1\rho_2\rho_3} = m_1 m_3';$$

and $\quad \Sigma g_1 D_{11} = m_1 \Sigma D_{11} - \Sigma(g_2 + g_3) D_{11}$. Now,

$$\Sigma(g_2 + g_3) D_{11} = \frac{\Sigma S\phi\theta\rho_1(\theta\phi\rho_2\theta\rho_3 + \theta\rho_2\theta\phi\rho_3)}{S\rho_1\rho_2\rho_3} = l_3(\phi\theta, \theta\phi, \theta).$$

In Art. 16, $l_3(\theta\phi, \phi\theta, \phi)$ has been reduced, and on further reduction, since

$$g_1(g_2^2 + g_3^2) = g_1(m_1 - g_1)^2 - 2m_3 = -m_1 g_1^2 + (m_1^2 - m_2) g_1 - m_3$$
$$= -m_1 g_2 g_3 - m_2 g_1 + (m_1 m_2 - m_3),$$
$$\Sigma A_{11} g_1 (g_2^2 + g_3^2) = -m_1 M_2 - m_2 l_3' + (m_1 m_2 - m_3) m_2'.$$

Also, by Art. 6, $2m_3(\Sigma a_{11})^2 - \Sigma a_{11} g_1 \Sigma a_{11} g_2 g_3 = 2m_3 m_1'^2 - (m_1 m_1' - l_2) l_3$, and so on interchanging θ and ϕ,

$$\Sigma D_{11} g_1 = -m_3'(m_1^2 - m_2) - m_1' m_2' m_2 + m_1 m_1' l_3' + m_2' l_3 - l_2 l_3' + m_1' M_2.$$

Hence, any invariant of the type $\Sigma D_{11} f(g_1)$ may be reduced to the ten fundamental invariants m_1, m_2, m_3, m_1', m_2', m_3', l_2, l_3, l_3', M_2, and the new skew invariant $N_3' - N_3$.

21.—*Second reducing system likewise depending on it.*

Denoting also the determinant $\begin{vmatrix} g_2g_3a_{11} & g_3g_1a_{12} & g_1g_2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ by the symbol E_{11}, $N_3' = \Sigma E_{11}g_1$.

$$E_{11} = \frac{S\,\theta\rho_1\,\phi\theta\rho_2\,\phi\theta\rho_3}{S\,\rho_1\rho_2\rho_3} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ g_1a_{21} & g_2a_{22} & g_3a_{23} \\ g_1a_{31} & g_2a_{32} & g_3a_{33} \end{vmatrix},$$ and therefore

$$\Sigma E_{11} = l_3(\phi\theta, \phi\theta, \theta) = m_2m_3'.$$

$$\begin{aligned} \Sigma E_{11}g_2g_3 &= \Sigma g_2g_3(g_2g_3a_{11}A_{11} + g_3g_1a_{12}A_{12} + g_1g_2a_{13}A_{13}) \\ &= \Sigma a_{11}A_{11}g_2^2g_3^2 + m_3\Sigma g_1(a_{23}A_{23} + a_{32}A_{32}) \\ &= \Sigma a_{11}A_{11}g_2^2g_3^2 + m_3[m_1m_3' - m_1\Sigma a_{11}A_{11} + 2\Sigma a_{11}A_{11}g_1]. \end{aligned}$$

by the formulæ of the third Article.

Now, $\quad g_2^2g_3^2 - m_1m_3 + 2g_1m_3 = g_2g_3(g_1-g_2)(g_1-g_3)$, and

$$\Sigma E_{11}g_2g_3 = \Sigma a_{11}A_{11}g_2g_3(g_1-g_2)(g_1-g_3) + m_3m_1m_3'$$

is reducible by the aid of the K functions of Art. 13, to a function of the ten fundamental invariants.

By the aid of this reducing system, any invariant of the form $\Sigma E_{11}f(g_1)$ is expressible in terms of the eleven invariants.

22.—*Reduction of invariants of derived functions.*

Take any function ψ for which $\psi\rho_1 = c_{11}\rho_1 + c_{12}\rho_2 + c_{13}\rho_3$ and combine it with θ and ϕ, so as to get invariants linear in θ and ψ,

$$\begin{aligned} l_2(\theta, \psi) &= \Sigma(a_{22}c_{33} + a_{33}c_{22} - a_{23}c_{32} - a_{32}c_{23}), \\ l_2(\theta\phi, \psi) &= \Sigma(g_2a_{22}c_{33} + g_3a_{33}c_{22} - g_2a_{23}c_{32} - g_3a_{32}c_{23}), \\ l_2(\phi\theta, \psi) &= \Sigma(g_2a_{22}c_{33} + g_3a_{33}c_{22} - g_3a_{23}c_{32} - g_2a_{32}c_{23}), \\ l_3(\phi, \theta, \psi) &= \Sigma g_1(a_{22}c_{33} + a_{33}c_{22} - a_{23}c_{32} - a_{32}c_{23}); \end{aligned}$$

and, therefore,

$$m_1 l_2(\theta, \psi) - l_3(\phi, \theta, \psi) = \Sigma(g_2 + g_3)(a_{22}c_{33} + a_{33}c_{22} - a_{23}c_{32} - a_{32}c_{23}).$$

Hence,

$$m_1 l_2(\theta, \psi) - l_3(\phi, \theta, \psi) - l_2(\theta\phi, \psi)$$
$$= \Sigma(g_3 a_{22}c_{33} + g_2 a_{33}c_{22} - g_3 a_{23}c_{32} - g_2 a_{32}c_{23}) = l_2(\psi\phi, \theta),$$

by interchanging the symbols c and a.

In like manner,

$$m_1 l_2(\theta, \psi) - l_3(\phi, \theta, \psi) - l_2(\phi\theta, \psi) = l_2(\phi\psi, \theta).$$

Again, interchanging θ and ψ, $l_2(\psi\theta, \phi)$, and $l_2(\theta\psi, \phi)$, are reduced to invariants of ψ, θ, and ϕ.

Also, $m_1(\psi\phi) = m_1(\phi\psi) = m_1 m_1(\psi) - l_2(\psi, \phi)$, with a similar result for $\psi\theta$ and $\theta\psi$.

23.—*Indication of more general reduction.*

More generally, if any invariant exists of the form $I = \Sigma \dfrac{S\,\phi\psi a\,\lambda\beta\,\mu\gamma}{S\,a\beta\gamma}$, where λ and μ are linear vector functions compounded in any manner of θ and ϕ,

$$I = \Sigma \frac{S\,\psi a\,\phi' V\lambda\beta\,\mu\gamma}{S\,a\beta\gamma} = m_3 \Sigma \frac{S\,\psi a\,V\phi^{-1}\lambda\cdot\beta\,\phi^{-1}\mu\cdot\gamma}{S\,a\beta\gamma}.$$

In like manner $J = \Sigma \dfrac{S\,\lambda a\,\phi\psi\,\beta\,\phi\psi\,\gamma}{S\,a\beta\gamma}$ may be written in the form

$$J = m_3 \Sigma \frac{S\,\lambda a\,\phi'^{-1} V\psi\beta\psi\gamma}{S\,a\beta\gamma} = m_3 \Sigma \frac{S\,\phi^{-1}\lambda a\,V\psi\beta\psi\gamma}{S\,a\beta\gamma}.$$

Hence, any invariant of $\phi\psi$, and of linear vector functions derived from θ and ϕ, is expressible in terms of invariants of ψ and other functions of θ and ϕ. So, if ψ itself is a function of θ and ϕ, and if the invariants of ψ and combinations of θ and ϕ are reducible to a common system, the invariants of $\phi\psi$ and combinations of θ and ϕ are reducible to the same common system. Similar remarks apply to $\theta\psi$.

24.—*On cycles of functions.*

Functions of the second order have already been considered. The functions of the third order are

$$\phi^3,\ \phi^2\theta,\ \phi\theta\phi,\ \theta\phi^2,\ \theta^2\phi,\ \theta\phi\theta,\ \phi\theta^2, \text{ and } \theta^3.$$

Of these, θ^3 and ϕ^3 are reducible by their cubics, and may be left out of account.

As in Art. 7, $\phi^2\theta$, $\phi\theta\phi$, and $\theta\phi^2$ satisfy the same cubic. Its coefficients are reducible to fundamental invariants,

$$m_1(\theta\phi^2) = \Sigma\, a_{11}g_1^2 = l_3 - m_1 l_2 + (m_1^2 - m_2)\, m_1'\,; \text{ by Art. 6};$$

$$m_2(\theta\phi^2) = \Sigma\, A_{11}g_2^2 g_3^2 = m_3 l_3' - m_1 m_3 m_2' + m_2 M_2, \text{ (cf. Art. 9)};$$

and $$m_3(\theta\phi^2) = m_3^2 m_3'.$$

By interchange of θ and ϕ, the coefficients of the cubic of $\theta^2\phi$, $\theta\phi\theta$, and $\phi\theta^2$ are found.

The functions of the fourth order are of the form $\theta\psi_3$ or $\phi\psi_3$, where ψ_3 is a function of the third order. There are ten of these functions,

$$\phi^2\theta\phi,\ \phi^2\theta^2,\ \phi\theta\phi^2,\ \phi\theta\phi\theta,\ \phi\theta^2\phi,\ \theta\phi^2\theta,\ \theta\phi\theta\phi,\ \theta\phi\theta^2,\ \theta^2\phi^2, \text{ and } \theta^2\phi\theta\,;$$

the obviously reducible functions not being counted. Of these, the first and third have the same cubic as $\phi^3\theta$ or $\theta\phi^3$; the second, by cyclical transposition of the symbols, is changed into the sixth, ninth, and fifth, and these have a common cubic; the fourth and seventh have a common cubic; and so have the eighth and tenth. The system of functions of the fourth order consists, therefore, of four cyclical groups; each group has a cubic common to the functions of the group; the coefficients of each of these four cubics are obviously reducible. In order, the cubics of these groups are the cubics of $\phi^3\theta$, $\phi^2\theta^2$, $(\phi\theta)^2$, and $\phi\theta^3$.

Of the fifth order, the cyclical groups are $\phi^4\theta$, $\phi^3\theta^2$, $\phi^2\theta\phi\theta$, $\phi\theta\phi\theta^2$, $\phi^2\theta^3$, and $\phi\theta^4$. The coefficients of the cubics of these functions are all expressible in terms of the ten invariants. For example, $\phi^2\theta\phi\theta = \phi(\phi\theta)^2$, and the reduction is similar to that used for $\theta\phi^2$.

This process seems to be capable of indefinite extension. Hence, it would follow that no function of the nth order can have any sequence

of symbols repeated three times without being reducible. For instance, $\theta(\theta\phi)^3\theta$ is reducible; $\theta(\theta\phi^2)^3\theta$ is likewise reducible, and so is $(\theta^2\phi^2)^3\theta$.

As regards cyclical groups, no irreducible function belongs to the cycle $\phi^5\theta$. The function $\phi^2\theta\phi^2$ is the only irreducible function of the cycle $\phi^4\theta$.

No cycle can have two sequences thrice consecutively repeated; no irreducible function is derivable from the cycle $\theta^3\psi\cdot(\theta\phi)^3\chi$.

25.—*Differential operators. Three functions.*

Arts. 11 and 22 indicate methods of dealing with the invariants of three linear vector functions.

Forming the cubic of $\phi + u\theta + v\psi$, the invariant $l_3(\phi, \theta, \psi)$ is obtained. In the notation of the Articles cited, from any invariant of θ and ϕ, an invariant connecting the three functions may be found by means of the differential operator $\Sigma c\frac{d}{da}$, which replaces a symbol a by the corresponding symbol c. On substituting $\theta + v\psi$ for θ in an invariant, this is evident. Thus, from the skew invariant two distinct invariants involving ϕ, θ, and ψ are determined by this operator. These are interchangeable on interchange of θ and ψ.

From an invariant of θ and ϕ, other invariants of the two functions may be obtained by the operator $g_1\frac{d}{da_{11}} + g_2\frac{d}{da_{22}} + g_3\frac{d}{da_{33}}$. Thus, from m_3' is derived $\Sigma A_{11}g_1$, and from this again, $\Sigma a_{11}g_2g_3$, and again, m_3.

It may be gathered from Art. 22, or from the equations,

$$\theta\phi\rho_1 = g_1(a_{11}\rho_1 + a_{12}\rho_2 + a_{13}\rho_3), \text{ and } \phi\theta\rho_1 = (g_1a_{11}\rho_1 + g_2a_{12}\rho_2 + g_3a_{13}\rho_3),$$

and others symmetrical with these, that the operator

$$\Sigma g_1\left(c_{11}\frac{d}{da_{11}} + c_{12}\frac{d}{da_{12}} + c_{13}\frac{d}{da_{13}}\right)$$

leads from an invariant θ and ϕ to an invariant of $\psi\phi$, θ, and ϕ; and also that the operator $\Sigma g_1\left(c_{11}\frac{d}{da_{11}} + c_{21}\frac{d}{da_{21}} + c_{31}\frac{d}{da_{31}}\right)$ leads to an invariant of $\phi\psi$, θ, and ϕ.